STAT 545 Homework 4

Quan Zhou*

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1. **Problem 3.32** (3 points)
First, let $\hat{\pi}_{ij} = n_{ij}/n$ be the MLE of π_{ij}. Each cell can be viewed as the sum of n i.i.d Bernoulli random variables, which we denote by $\xi_{ij}^k (k = 1, \ldots, n)$. Obviously $E[\xi_{ij}] = \pi_{ij}$. Moreover, for any two distinct cells,
$$Cov(\xi_{ij}, \xi_{ab}) = E[\xi_{ij}\xi_{ab}] - \pi_{ij}\pi_{ab} = -\pi_{ij}\pi_{ab}$$
since at each trial only one cell could be selected. By Central Limit Theorem,
$$\sqrt{n}(\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}) \xrightarrow{D} \mathcal{N}(\mathbf{0}, Diag(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}')$$
where $\boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$. Let $\boldsymbol{\theta = (\nu, \delta)}$. Recall that a function of MLE is still an MLE, then by Delta method,
$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{D} N(0, D'_{\boldsymbol{\pi}}\boldsymbol{\theta}(Diag(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}')D_{\boldsymbol{\pi}}\boldsymbol{\theta})$$
where $\hat{\boldsymbol{\theta}}$ is the MLE. Again by Delta method,
$$\sqrt{n}(\hat{\zeta} - \zeta) \xrightarrow{D} N(0, D'_{\boldsymbol{\theta}}\zeta D'_{\boldsymbol{\pi}}\boldsymbol{\theta}(Diag(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}')D_{\boldsymbol{\pi}}\boldsymbol{\theta}D_{\boldsymbol{\theta}}\zeta)$$
where $\hat{\zeta}$ is the MLE. To simplify the covariance matrix, we first notice that
$$\begin{aligned} D'_{\boldsymbol{\theta}}\zeta D'_{\boldsymbol{\pi}}\boldsymbol{\theta} &= (\frac{1}{\delta}, -\frac{\nu}{\delta^2})D'_{\boldsymbol{\pi}}\boldsymbol{\theta} \\ &= \frac{1}{\delta^2}(\delta\frac{d\nu}{d\boldsymbol{\pi}} - \nu\frac{d\delta}{d\boldsymbol{\pi}}) \end{aligned}$$
Now let $\eta_{ij} = \delta\frac{d\nu}{d\pi_{ij}} - \nu\frac{d\delta}{d\pi_{ij}}$. With a little algebra, the covariance matrix is
$$\begin{aligned} & D'_{\boldsymbol{\theta}}\zeta D'_{\boldsymbol{\pi}}\boldsymbol{\theta}(Diag(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}')D_{\boldsymbol{\pi}}\boldsymbol{\theta}D_{\boldsymbol{\theta}}\zeta \\ =& \frac{1}{\delta^4}\boldsymbol{\eta}'(Diag(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}')\boldsymbol{\eta} \\ =& \frac{1}{\delta^4}[\sum_i\sum_j \pi_{ij}(1-\pi_{ij})\eta_{ij}^2 - \sum\sum_{\text{distinct cells}} 2\pi_{ij}\pi_{ab}\eta_{ij}\eta_{ab}] \\ =& \frac{1}{\delta^4}[\sum_i\sum_j \pi_{ij}\eta_{ij}^2 - (\sum_i\sum_j \pi_{ij}\eta_{ij})^2] \end{aligned}$$

*an adapted version from Quan's original homework.